Explaining Mathematical Truths with a Switch of Structure

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Mathematicians ask why-questions, how-questions and what-questions. Why are the prime numbers distributed as they are? How can we characterise a three-dimensional sphere topologically? What *really* is isomorphism? Such questions demand explanations and, by doing so, influence the course of mathematical research. Such questions are asked in the purest and most abstract areas of mathematics, without concern for matters of the physical world. Such questions are sometimes answered *by* mathematics. In short, mathematicians are often concerned with the explanation .

Philosophers have long understood that explanation has an important role in science, but have perhaps too often assumed that an explanation of a mathematical truth comes ready-made with its proof. More recently, however, the role of explanation in mathematics has been identified as an interesting philosophical issue and comparisons have been made between mathematical and scientific explanation, most notably by Mancosu (2001) and Sandborg (1998). Instead of looking to accounts of scientific explanation, however, I complement this work by calling upon established philosophies of mathematics to answer the question before us: how can mathematics explain itself?

In the first section of this paper I provide new examples to demonstrate that proof is an insufficient criterion of mathematical explanation. This exercise also provides clues to where we may find answers to our riddle. In the second section, I survey traditional viewpoints of mathematics to see how they account for mathematical explanation. I find that the best hint of an answer emerges from structuralism and I sketch this response. Finally, in the third section, I give motivation for approaching the question of mathematical explanation in the way that I have done.

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Because of its specific nature, I suspect that in this case it is the mathematical that will shed light on the scientific and hence I conclude that mathematical explanation is potentially a rewarding avenue for philosophers to pursue.

Before we get to work, let me make two clarifications. First, in suggesting that there are occasions when mathematicians provide explanations, I do not mean that mathematics provides explanations for scientific phenomena, although of course it does. I wish to discuss cases when *the explanandum itself* is a mathematical truth. Secondly, I do not wish to make the claim that every time a mathematician asks a why-question (or any other interesting question) that this should necessarily be catered for by a theory of mathematical explanation. Sometimes mathematicians appear to be asking for explanation when they are simply seeking proof. I only urge that we explicate the concept of mathematical explanation and that it should agree with the intuitions of mathematicians to a certain (even limited) extent.

I: There is explanation in mathematics, and it is not merely a matter of deduction

A mathematical truth may prompt a mathematician to seek an explanation which is not merely a logical deduction from axioms. This is best shown with examples.

(Ex.1) The Galois explanation: why certain classes of equations have solutions which are expressible in radicals (and others do not)

A radical is a mathematical expression composed of the coefficients of an equation using only the operations of addition, subtraction, multiplication, division and taking nth roots (square roots, cube roots etc). Until the work of Galois in the 1830s, it was a curious fact that the solutions to polynomials of second, third and fourth degree can be given in radicals. For example, the quadratic polynomial

$$ax^2 + bx + c$$

has two roots given by the radical

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad [1]$$

Moreover, it was proved by Ruffini in 1799 and Abel in 1824 that it is not possible to give a general solution of this sort for the polynomial of fifth degree. Here is our why-question: why is there no radical solution to the fifth-order polynomial?

It turns out that any polynomial can be associated with a group by considering the symmetries of its roots. Galois reaped the rewards of converting questions regarding polynomials and fields to easier ones about symmetry groups. In particular, he demonstrated that it is a particular property of symmetry groups which determines whether the solution to the related class of equations can be written in radicals. Galois Theory can be viewed (and is viewed by mathematicians) as an *explanation* as to why the quadratic, cubic and quartic polynomial have such radical solutions, but polynomials of higher powers do not. The theory even helps us to see why [1] takes the form it does, relying on the symmetry of the quadratic roots.

In comparison to the proof given by Galois Theory, there is a basic proof of [1] which depends solely upon elementary, high-school mathematics. This straightforward, arithmetical calculation, however, does not explain. Similarly, Abel's proof that the polynomial of fifth degree does not have solutions expressible in radicals doesn't make the grade either. What is special about the proof of Galois? Does a greater understanding of mathematics *always* arise when a problem in one area (fields) is converted to one in another (symmetry groups)?

(Ex.2) Borcherd's Explanation: Why infinite Lie algebras have some particular (and quite peculiar) properties

In the 1980s, Kac (1982) published a seminal work in the representation theory of infinite Lie algebras. Towards the end of this book, he proved a number of theorems, presented messily using vertex operators. The computations appeared strange to mathematicians at the time. The background of the mathematics is too complex to give here, but the main point is this: Kac did not know that he was working with vertex algebras. He could not explain his results, and their significance was not truly realised until Borcherds (1986) tidied up the notion of vertex algebra theory.[1] It is now impossible to work in the field without reference to vertex algebras, and already Kac's formulation of his results, although accurate, is obsolete. Vertex algebras are thought to provide a deeper understanding of representations of infinite Lie algebras, they *explain* Kac's computations. Again, one proof is favoured over another when compared for their explanatory value. But what is remarkable about Borcherd's reformulation of Kac's calculations? Do we *always* get better understanding of a piece of mathematics (a Lie algebra) when we encapsulate it in a different structure (a vertex algebra)?

In both of these cases, a proof of the mathematical result exists independently of the explanatory proof. Our exercise here is to identify which particular characteristics make one of them an explanation and not the other. In this search for a tech-

[1] Physicists had been playing with vertex algebras for years, and indeed donated the word 'vertex' to mathematicians. So it would be incorrect to say that Borcherds introduced vertex algebras. He was one of the first to uncover their mathematical structure formally.

nical description of explanation, I admit that I begin with the approximate notion that explanation improves understanding of mathematical objects. The cases under consideration then lead me to the conjecture that:

[Con] Proof is insufficient to account for explanation in mathematics. Mathematical explanation consists of an (at least partly) objective component in addition to its deductive nature.

In the next section, we will see how various philosophies of mathematics stand up to this conjecture. Meanwhile, I summarise the evidence for [Con] so far:

(E1) Mathematicians search for new proofs with better explanatory power.

(E2) Explanatory proofs have objective features which distinguish themselves from other proofs. In the two cases studies, it is an unexpected link to a different area of mathematics that provides fresh understanding.

(E3) It is unsatisfying that a statement should be self-explanatory, yet if proof and explanation are equated, a self-consistent statement (taken as an axiom) provides an explanation of itself.

(E4) Proofs have symmetries that explanations do not observe. Proofs can be reorganised to deduce statements which were originally taken to be axioms, but explanations can not always withstand such treatment.

So what does an explanation of a mathematical fact look like?

II. Mathematical explanation as a structure-switch

It is natural to turn first to established theories within the philosophy of mathematics for answers to our question. This investigation is merely an outline, but it is sufficient to show that there are no complete accounts of mathematical explanation consistent with [Con] readily available. There are, however, seeds of solutions lying around.

(1) Formalism: explanation is a historical record of symbol manipulation

Formalism holds that mathematics is the manipulation of symbols. On the most extreme version, there is no meaning to the symbols beyond the symbols themselves and the rules of manoeuvring. Another rendering of formalism allows logical symbols to be interpreted in the usual fashion, and mathematics is understood as the deduction of logical consequences from uninterpreted axioms. There are not many places to look for an account of explanation here. If we consider mathematics to be nothing more than symbols, then there isn't anything to explain about a mathematical statement because mathematics is meaningless. If, on the other hand, we are to accept that mathematics is the logical consequence of uninterpreted axioms, we can consider that a demand for explanation is a request to demonstrate how a mathematical statement was derived using the rules of manipulation, and from which uninterpreted axioms. At best, the understanding that mathematicians have of mathematical objects can be considered as something akin to the satisfaction of solving a crossword puzzle. This equates mathematical explanation with 'proof' (if a history of the manipulation of meaningless symbols passes as proof), contradicting [Con]. There is nothing to distinguish an explanatory proof from another.

Some formalists may wish to say that some mathematical axioms are interpreted in order to comply with a scientific theory, that this is in fact precisely what it is to apply mathematics to the world. This approach does not suit our purposes either. If we restrict explanation to proofs within interpreted systems, we have not satisfied the demand to provide an account of *mathematical* explanation. Once the axioms have been interpreted as scientific theory, we are left with an explanation of a *scientific* fact.

It is not a surprise to see formalism fail for the project in hand, for I am at the outset assuming that it is possible for mathematicians to gain understanding of their subject matter. For this to be true, mathematics has to be more than the symbols themselves that mathematicians write down. I am prepared to admit, therefore, that before I began I had shut the door to formalism.

(2) Logicism: explanation is logical deduction

Russell (1919) said there is nothing more to mathematics than logic. This differs from deductive formalism in the sense that mathematical objects may be understood to exist in their own right, as logical objects. One type of explanation that arises in logicism is when one area of mathematics is reduced to logic. In this way it could be claimed that Frege explained what arithmetic *is*. However, exercises of this kind are philosophical rather than mathematical. They do not explain a mathematical truth but rather an area of mathematics, attempting to put solid ground beneath the feet of mathematicians. This is not the type of explanation we are discussing presently.
When considering the explanation of a mathematical truth from the viewpoint of logicism, we must conclude that explanation is nothing more than a deductive proof. Explanation here is more substantial than within formalism, because an explanation can now properly be said to provide understanding. This gift, however, comes solely a consequence of the explanation being a proof. Clearly logicism is not rich enough to account for [Con]. If all mathematics is deductive logic, a mathematician can only prefer one proof over another for reasons of his own, of the community he works in, or of the human mind. It could be that he finds a particular proof more beautiful, or that his mathematical training makes him appreciate a particular way of proving statements, or that the wiring of the human brain means that some proofs provide

a deeper understanding of mathematical objects than others. However, there is no characteristic of logic itself that makes one piece of logic explain better than another.

(3) Platonism: an account of explanation is welcome but is yet to be found

Platonism holds that mathematics exists independently of the physical world as a realm of abstract and non-mental objects. Is there anything which looks like explanation in this world? My view is that an account of explanation does not naturally fall out of Platonism but that, as a result of this, Platonists are able accommodate many different approaches to mathematical explanation. They can, for example, welcome the proposal from logicism that explanation is nothing more than proof. It is also plausible that some proofs uncover previously unknown properties of mathematical objects and that this is the essence of explanation. It was always the case, for instance, that polynomials have properties relating to the symmetry group of their roots. Galois merely discovered and exploited this in his proof. Because Platonists tell us nothing about explanation in the first place, we are able to impose many theories of explanation upon it. I suggest therefore that Platonism is compatible with [Con] but doesn't provide many clues as to where to go next.

(4) Fictionalism: there is no explanation in mathematics

If all mathematical statements are vacuously true as a result of there being no mathematical objects to begin with, mathematical explanation cannot exist either. We cannot objectively explain fictional concepts and are left at the opposite extreme of logicism, for no deductive proof provides a mathematical explanation. A fictionalist may perhaps respond that mathematicians determine for themselves which are explanatory proofs, because this is not provided by the mathematics itself, but this will introduce an element of subjectivity contrary to [Con]. I conclude therefore that we cannot resolve our issue of mathematical explanation as fictionalists.

(5) Structuralism: the reply that explanation is a structure-switch

The slogan of structuralism is that mathematics is the science of structure. A structure is a collection of mathematical relations, and mathematical objects are the placeholders in these webs of relations (Shapiro, 2000). Mathematicians investigate the properties of structures in the same way that physicists investigate the properties of physical objects. In this setting, it is natural that mathematics should ask questions of a similar form, and seek explanations in a similar way to scientists. So structuralism provides some response to our question, at least the ante rem structuralism of Shapiro (1983) and Resnik (1981). Ante-remists insist that mathematical structures exist independently of the objects that exemplify them. A structure exists as an abstract object and as a result it exemplifies itself. Shapiro (2000) holds that the place-holders in the structure (the mathematical objects) therefore exist. He tells us that mathematical structures can exemplify themselves and other mathematical structures. His example is the Zermelo and von Neumann set constructions which both exemplify the structure of the natural numbers.

Perhaps Shapiro would agree that Galois' roots of polynomials exemplify group structures. Explanation here consists in demonstrating that one mathematical structure contains relations which exemplify another. I call this a 'structure switch'. In a similar vein, Borcherd's explanation showed that the strange-looking calculations of Kac are actually relations in a different structure, namely, that of vertex algebras. In both cases, the explanation involved a switch from working in one structure to working in another. To put this more clearly, if we have a mathematical truth T, supported by a collection of mathematical structures S, then an explanatory proof of T is one which demonstrates that a part of S exemplifies a structure S* where S* is not a part of S.

This is, of course, nothing more than a first, hazardous guess in the direction of mathematical explanation. It is to be technically worked out. What do we mean to say that one structure 'is not a part of' another, when it exemplifies the other? What is it for a mathematical truth to rely on a structure? What is it that makes the structures of groups and polynomials sufficiently unconnected to claim a switch of structure has occurred? Does the new structure always have to be exemplified by the original mathematics, or is there another way in which it can be used? My point is that structuralism provides a hint towards resolution; at a first glance it provides the most promising account of mathematical explanation. Note that the structure-switch theory of explanation does not allow a statement alone to be self-explanatory. Further, a reorganisation of an explanatory proof may or may not itself be an explanation and hence the asymmetry of explanation is honoured.

III. How mathematical explanation may shed light on science

There are so many varieties of explanation to be found in the world that it is difficult to repudiate a pluralist and somewhat subjective account of scientific explanation. Furthermore, it is not clear how these accounts can be successfully transported to the mathematical realm if [Con] is to be respected. If we apply the deductive-nomological model to mathematics, for example, we are forced to equate mathematical explanation with proof. The reverse may be more rewarding, for we have a cleaner problem to deal with in the mathematical world. After all, explanation is context-dependent and in mathematics context is easier to identify: it is simply the total mathematical knowledge behind a truth T.

So, are there explanations of physical phenomena which are based upon a structure switch? I think that there are. Consider the increased understanding of the electromagnetic, weak and strong forces that came with the recognition of gauge symmetries. Physicists noticed that the concepts they were working with exemplified particular mathematical structures and they were able to *explain* phenomena using features of these new structures. The gauge symmetries indicated why apparently diverse forces were different aspects of the same thing. Unification theories like those of Kitcher (1981) assert that the explanatory power of these stories results from subsuming basic laws under more general laws. I suggest instead that the unification of physical forces was a result of a structure switch which occurred in the background mathematical theory and which provided the explanation the physicists sought.

I admit this has been an introductory sketch, doing nothing more than highlighting a question and suggesting where there is interesting work to be done. I say there is evidence to suggest that explanation and proof are not identical in mathematics. I see hope that a solution can be found in the structure-switching that appears in fields as diverse as Galois theory and the unification of physical forces. Mathematical explanation promises to be a tidier problem than its scientific counterpart and there are signs that some of its results might translate nicely into the scientific realm. For all these reasons, I suggest that mathematical explanation is in need of some proper explanation.

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