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Analytical Approximation of the Results of Schelling's Checkerboard Model

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Abstract

Thomas Schelling shows in his checkerboard model that even mildly segregationist preferences of the single agents are sufficient to bring about high rates of segregation. This paper analytically investigates the model. Taking advantage of the knowledge about the agents' preferences, a notion of available neighbourhood compositions (those fulfilling an agent's preferences) is developed. The unweighted average segregation of those available neighbourhoods allows a reasonable approximation of the simulation results, which overrates the actual final rate of segregation instead of underrating it as done intuitively. Additionally, the concept of neighbourhood compositions possibly serves as explanations for the behaviour of the simulation.

Introduction

The importance of Schelling's spatial proximity model or 'checkerboard model' for the growth of social modelling as best practice example is uncontrover-

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sial.[1] It undoubtedly provides a powerful tool to tackle the problems in its fields of application and changed the way in which those problems were perceived. As Zhang puts it: 'The spatial proximity model has every ingredient of a great theoretical work: it addresses an important real world issue; it is simple to understand; it produces unexpected results; and it has deep implications for various social sciences.'[2]

Schelling envisions an 'area' either a line or a grid of cells,[3] populated by a fixed number of agents of two 'kinds'. When applied to residential segregation, those kinds are black/white but there are several other interpretations such as boys/girls or poor/rich. Agents have preferences over the kind of their neighbours and move starting from a patterned or more often random initial distribution to fulfil these preferences. At the time Schelling proposed his models, most authors thought that the empirically observed residential segregation was brought about by highly segregationist preferences or very

[1] It is wrong to give credit for this model to (solely) Schelling, because actually 'James M. Sakoda was the first person to develop a CA [cellular automata, A/N] based model in the social sciences. The model did not see the light of the day as part of the CA literature at the time or make any reference to previous CA research. It was baptised a 'checkerboard model'. Sakoda published the article The Checkerboard Model of Social Interaction in 1971 but the basic design of the model was already present in his unpublished dissertation of 1949. The central goal of his model was to understand group formation. The model 'grew out of an attempt to portray the interaction in a relocation center during World War II' (Sakoda 1971, p. 120), where members of the Japanese minority in the US were evacuated after Japan's attack on Pearl Harbour.' (Hegselmann and Flache, 'Understanding Complex Social Dynamics: A Plea For Cellular Automata Based Modelling'). While offering a broader approach to modelling group formation processes, Sakoda's model is able to explain residential segregation as Schelling does (Sakoda, 'The checkerboard model of social interaction'). However, this paper will refer to Schelling, since his model is the one famously worked with in the literature.

[2] Zhang, 'Tipping and Residential Segregation: A Unified Schelling Model'.

[3] Note here: Pancs and Vriend have shown that the one-dimensional and the two-dimensional model are not two versions of the same model but two distinct models. (Pancs and Vriend, *Schelling's Spatial Proximity Model of Segregation Revisited*.

restrictive social institutions. Schelling finds that, contra this widespread position, even moderate preferences for same-kind neighbours at the individual level are likely amplified into high segregation at the macro level.[4] Therefore, macrobehaviour in a society does not represent its individual members' micromotives.[5]

This paper investigates agents' preferences analytically to work out a concept of neighbourhood compositions, which agents with a certain preference are happy within. Using that concept, it is possible to analytically approximate the simulation results, i.e. only using data given directly in the model description. Such a prediction questions the emergence of Schelling's simulation results and their importance for showing that segregation comes about. However, the concept developed in this paper may help to uncover the mechanisms at work in the checkerboard model, too. Thus, it enables to use simulation to understand why and how segregation comes about.

I. Description of the Checkerboard Model Underlying this Paper

The size of the two-dimensional model world is 21 by 21 cells. It has the shape of a torus (despite the finite size of the world there are no borders and corners), thus each cell has eight direct neighbouring cells. Seventy-five percent of the cells are occupied, so the population is 330, the two kinds having equal numbers. The definition of the neighbourhood of an agent is taken from Moore, i.e. each agent has a maximum of eight neighbours. When the satisfaction of an agent's preferences is checked, she counts herself as a neighbour of her own kind; empty cells in the neighbourhood are not counted at all.

The agents' preferences, taken from Schelling, are represented by the following utility function:

$$u_i = \begin{cases} 0 & \text{for } \frac{x_i}{y_i} < \theta \\ 1 & \text{for } \frac{x_i}{y_i} \geq \theta \end{cases}$$

[4] Schelling, 'Models of Segregation'.

[5] Schelling, *Micromotives and Macrobehavior*.

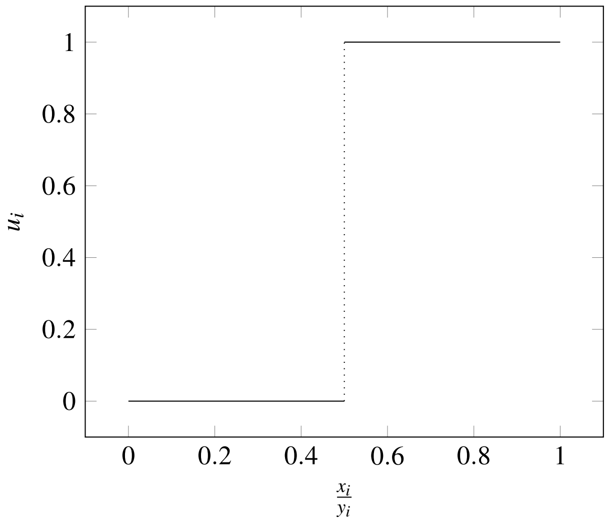

Figure 1: Utility function of the agents with $\theta = 0.5$

Here, y is the total number of agent i's neighbours and x is the number of agent i's neighbours being of the same kind as i. If the ratio is below a certain threshold θ, the agent is discontent. There is no additional (random or pre-set) factor influencing the calculation whether an agent is content or not. Figure 1 depicts the utility function.

At the start of the simulation, the agents are distributed randomly. Agents are randomly picked for a possible movement, but only discontent agents actually move. They do so to the closest free cell satisfying their preferences. Distance between two cells is the smallest number of cells that one has to pass (horizontally and vertically) in order to get from one cell to the other. If there are two or more cells satisfying an agent's preferences in exactly the same distance, there is a random selection between them, if there is none in the whole world, the agent stays where she is. By moving, an agent may make previously unhappy agents in her old or new neighbourhood happy and vice versa. This mutual influence of agents is not determinable analytically.

II. Analytical Remarks Concerning Neighbourhoods

It is possible to classify the neighbourhoods by considering only those features relevant for the agents' decision whether they are happy. This means, neighbourhoods differ in the number of total and own-kind neighbours living within (since those are necessary to calculate the rate of own-kind neighbours). Such a classification yields 45 different neighbourhood compositions, all bringing about a certain rate of own-colour neighbours, shown in Table 1.[6]

	1	2	3	4	5	6	7	8	9
1	1.00	0.50	0.33	0.25	0.20	0.17	0.14	0.13	0.11
2		1.00	0.67	0.50	0.40	0.33	0.29	0.25	0.22
3			1.00	0.75	0.60	0.50	0.43	0.38	0.33
4				1.00	0.80	0.67	0.57	0.50	0.44
5					1.00	0.83	0.71	0.63	0.56
6						1.00	0.86	0.75	0.67
7							1.00	0.88	0.78
8								1.00	0.89
9									1.00

Table 1: Neighborhood segregation (number of own-kind, vertical) depending on the number of agents in the neighbourhood (horizontal)

This rate of own-colour neighbours will be called neighbourhood segrega-

6 Schelling himself is aware of an agent's possible neighbourhoods but does not expand on their potential importance to explain the phenomena observed by him. A possible reason is that he avoids relying solely on computational evaluation (Schelling, 'Dynamic Models Of Segregation'), even though being willing to 'Letting a Computer Help with the Work.' (Schelling, 'On Letting the Computer Help with the Work - Teaching Material'), as shown by Hegselmann (Hegselmann, 'Thomas C. Schelling and the Computer: Some Notes on Schelling's Essay "On Letting a Computer Help with the Work".')

tion. By comparing θ to the neighbourhood segregations, one can determine how many neighbourhood compositions are available to agents with a certain threshold. For θ-values as in Table 2, this reveals a continuous – almost linear – decrease of the number of available neighbourhood compositions with θ becoming more demanding, which is not surprising.

θ	Neighbourhoods
0.1	45
0.2	41
0.3	36
0.4	32
0.5	29
0.6	23
0.7	18
0.8	14
0.9	9

Table 2: Segregationist preference and number of available neighborhoods

However, the number of available neighbourhood compositions alone does not allow establishing how hard it is for an agent to find a neighbourhood satisfying her preferences, because certainly not all neighbourhoods are equally likely to occur and the likeliness of occurrence at a certain point in a simulation run is highly dependent on the previous history of that run: under random distribution, it is trivial to calculate how often a certain neighbourhood occurs in average (e.g. due to the population specifications a neighbourhood will most likely have 7 agents in it, 3 or 4 of them having the colour of the agent in question). As purposive movements of agents come into play, such calculations are not possible, since an agent's movement influences the availability of neighbourhoods for all other agents. This mutual influence of agents' movements on the available neighbourhoods (and thus the satisfaction of their preferences) seems to cause the emergent dynamic of the simulation.

Availability of neighbourhood compositions is determinable by purely analytical means because it follows directly from properties of the agents, stated explicitly in the model description. The emergent effect in a checkerboard model is a process of segregation resulting in a state not expected when analysing the available neighbourhood compositions of an individual agent. Therefore, on the one hand we should not expect the simulation results to be predictable by an analysis of the available neighbourhood compositions. On the other hand, available neighbourhood compositions are a possible key to understanding why and how segregation comes about.

III. Simulation Results

The simulations show that the checkerboard model with segregationists yields the results described by Schelling and proven robust by several authors, as Muldoon et al[7] or Zhang[8]. Nevertheless, the question is, in which sense those results are emergent. The final rate of segregation at a macro level is indeed much higher than the agents' individual threshold values for own-kind neighbours. However, as explained above, an analysis of agents' preference is possible by looking at their available neighbourhood compositions, too.

Hence, the simulation results at a macro level should as well differ from any simple aggregation of available neighbourhood compositions, which does not take into account the interdependences of agents' movements mentioned above. Clearly, it is such an aggregation to calculate the unweighted average segregation of available neighbourhood compositions for a given θ: It is achieved by listing all available neighbourhoods for an agent with a preference of θ, calculating the ratio of own kind agents in each neighbourhood and getting the average of all ratios. This average segregation of available neighbourhoods will be called Θ because it is analytically calculated from the threshold value θ.

[7] Muldoon, Smith and Weisberg, 'Segregation That No One Seeks'.

[8] Zhang, 'Residential Segregation in an All-Integrationaist World'.

θ	Segregation	ϑ	Difference
0.1	0.570	0.600	-0.030
0.2	0.577	0.645	-0.069
0.3	0.608	0.701	-0.094
0.4	0.677	0.756	-0.069
0.5	0.744	0.779	-0.0354
0.6	0.883	0.847	0.036
0.7	0.954	0.903	0.051
0.8	0.700	0.947	-0.247
0.9	0.582	1.000	-0.4 18

Table 3: Comparison of analytical and simulative results

Table 3 contrasts Θ with the simulation results for a certain θ. It shows that Θ allows a reasonable prediction of the simulation results, as long as there is a strict equilibrium tendency.[9] Furthermore, for mildly segregationist preferences, the results are overestimated. The simulation results are considered emergent because they are underestimated when using θ to predict segregation. Thus, analysing available neighbourhoods to predict the simulation results more accurately questions their emergence.[10]

Nevertheless, there still is a surprising result, namely the difference between the average segregation of available neighbourhoods and average rate of segregation resulting from the simulation. This emergent feature has to be explained by further investigating the simulation and working out mechanisms. It was shown that relying on Θ instead of θ yields reasonable predictions about the final rate of segregation but jury is still out, whether relying

[9] Note here, that due to the low variance of final segregation, not only average values but the result of a single run, too, are predictable.

[10] These findings are not solely caused by the neighbourhood definition underlying this paper, i.e. similar (and partly even stronger) results are generated when agents do not count themselves as their own neighbours, as the appended data shows.

on Θ to make statements about the agents' behaviour and thus the simulation results is reasonable itself.

IV. Reasons for Making Use of the Average Neighbourhood Segregation to Evaluate the Model and Predict its Results

Speaking of some situation m^* as preferred by an agent i often perceives that the agent maximises her utility, if and only if she is places herself in m^*: Speaking of some situation m^* as preferred by an agent i often perceives that the agent maximises her utility, if and only if she places herself in m^*:

$$\forall m' \neq m^* : u_i(m') < u_i(m^*)$$

Furthermore, if situations only differ with respect to the amount of a good X, the agent maximises her utility by getting the amount of X that she gets in m^*:

$$\forall m' \neq m^* : x(m') \prec_i x(m^*)$$

Thus, someone first looking at the simulation of the checkerboard model for agents with mildly segregationists preferences is surprised about the highly segregated result not because those results could not have been foreseen but because she implicitly dropped the 'at least' when interpreting the description 'a preference for at least θ own kind neighbours'.

However, θ is the neighbourhood segregation, which an agent is willing to exceed in order to maximise her utility. In the checkerboard model, good X is the neighbourhood segregation: $\frac{x_i}{y_i} = x$. This means, the amount of X in the preferred situation $x(m^*)$ is exactly the threshold value θ:

$$x(m^*) = \theta$$

From the agent's utility function, it follows that while neighbourhoods bringing about a lower rate of neighbourhood segregation than θ give the agent less utility, all neighbourhoods bringing about a rate of neighbourhood segregation of θ or higher than θ give the agent the same utility:

$$\forall m' \text{with} x(m') < \theta : u_i(m') < u_i(m^*)$$

$$\text{But: } \forall m' \text{with} x(m') > \theta : x(m') \sim_i x(m^*)$$

$$\text{Thus: } \forall m' \text{with} \frac{x_i}{y_i} > \theta : u_i(m') = u_i(m^*)$$

This means, θ is no suitable choice to start an evaluation of the simulation process and results with. Available neighbourhoods in general better capture, when an agent's preference is fulfilled. When analytically examining the model, one knows, which neighbourhoods are available to segregationists and which rate of segregation they would bring about when taken. However, one does neither know, how often these neighbourhoods will occur nor how often they will be taken in an equilibrium stable state. In that situation, it is a common strategy to assume an equal or normal distribution of actually taken neighbourhoods over the set of available neighbourhoods. One may then approximate the mean neighbourhood segregation of the taken neighbourhoods (which equals the final rate of segregation on the macro level in the simulation) by calculating the mean neighbourhood segregation of available neighbourhoods. That means, it is neither a non-reasonable nor an uncommon idea to use Θ when describing or predicting agents' behaviour , but several refinements and enrichments e.g. by taking into account parameters of dispersion are possible. By way of illustration, one may calculate a $\Theta' < \Theta$ by firstly applying a normal distribution of the agents over all neighbourhoods and after that normally redistributing all agents initially placed on unavailable neighbourhoods over all available neighbourhoods. Since Θ' captures the notion of an 'expected movement behaviour' of (initially) unhappy agents not taking into account interactions, any differences between Θ' and the simulation output are likely caused by agents' movements influencing other agents. Alternatively, it is a calculation of lower bounds for the final rate of segregation given a certain θ using methods of linear optimisation is worth considering. In general, Θ seems to be one key to capture the behaviour of the model.

Conclusion

In this essay, I argued that it is possible to predict the simulation output of the checkerboard model with segregationists only using data directly derived from the model description and thus independent from simulating the model. To do so, one simply has to take advantage of the knowledge about the agents' preferences and the neighbourhood definition used in the model to grasp the notion of available neighbourhood compositions. Available neighbourhood compositions are those, with a rate of own kind neighbours of at least the agents' threshold value. With that information it is possible to calculate the average segregation of available neighbourhoods and use it as reasonable prediction of the simulation results concerning segregation, as long as there is a strict equilibrium tendency. Moreover, using the average segregation of available neighbourhoods instead of the threshold value as a starting point, the simulation results in a rate of segregation lower than expected for mildly and moderately segregationist preferences. Thus, high segregation caused by low threshold values does not seem to be a surprising effect. Additionally, the consideration of available neighbourhood compositions points towards the necessity of explaining the mechanisms at work during a segregating process in the model and may provide a tool to do so, especially when supported by calculation of lower bounds.

Appendix

The appendix containing all the simulation outputs, further information about the evaluation and the model (with its ODD-description) used to obtain the benchmark results for this paper is available to the reader electronically:

https://www.dropbox.com/sh/6hmrae4dqwlh4uw/AABi9VmU6sceDdRRheiy Mb85a?dl=0

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